Ch0 - Math Recap - 1

(*Sets and Related Ideas*)

ECO 104 - Statistics For Business and Economics - I

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Outline

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Regarding Math Recap

We need some Math!

1. We will need some math in this course, so first we will do a quick recap!
2. This is going to be Math Recap Part 1, where we will cover mainly *sets*
3. In Part 2 Recap we will we will cover *functions, and some counting methods!*, but that will come before Probability theory in Chapter 2.

But don't worry!

1. These math topics should be familiar to you.
2. Most of you have already studied College-Level math (H.S.C Math or A Level Math) or at least took Math 100 at EWU.

But still....

1. If you feel a bit rusty with these concepts, please take the time to go through the material.
2. Note that this recap is not a replacement for a full math course but is intended to refresh your knowledge on essential topics.
3. So let's start with the recap!

1. **Sets**
   - a. Definitions and Examples
   - b. Set Operations
   - c. Different Sets of Numbers

2. **Functions and Related Ideas**

3. **Math Recap - Counting Methods**
   - a. Multiplication Rule
   - b. Permutation
   - c. Combination

Sets

a. Definitions and Examples

Definition 2A.1:(Set)

A set is a *collection of objects* treated as a *single* entity, the members are called *elements*. If X is a set and x is an element, we write $x \in X$, and this reads as "x belongs to X".

- Note the symbol $\in$ means "belongs to". For example, think about the set of even numbers between 1 and 11, if we *enumerate* then we can write this set as $S = \{2, 4, 6, 8, 10\}$. Note in this case, $2, 4, 6, 8, 10$ are all elements of the same set, so we can write $2 \in S$, $4 \in S$, and so on...

- Now in math we can write the same set also as,

$$S = \{x : x \text{ is an even number between 1 and 11}\}$$

- Where this means *"S is a set of element x, such that x is an even number between 1 and 11"*, note the symbol ":" means *"such that"*.

- Note we wrote the same set in two different ways,
- The first method is called *enumeration method*
- And the second one is called *set builder method*.
- Sometimes you will also see a slightly different notation "|" instead of ":". For example, we can write the same set as

$$S = \{x \mid x \text{ is an even number between 1 and 11}\}$$

- Question: What's the benefit of set builder method?

- Now we will see some quick definitions related to set.
- *Empty Set:* When a set has no element, then we call this an *empty set*. This set is denoted by $\emptyset$ or $\{\}$.
- *Equal Sets:* Two sets X and Y are equal if they contain exactly the same elements. and we write $X = Y$
- *Subset:* If we have two sets X and Y, and all the elements of a set X are also elements of the set Y, then X is a called a *subset* of Y, and we use the notation $X \subset Y$. Note that in this case Y is also called a *superset* of X.
- *Proper Subset:* Note when we write $X \subset Y$, then the set X may have exactly same elements as Y or may have less than Y. If all the elements in set X are in a set Y, but not all the elements of Y are in X, then X is called a *proper subset* of Y. In this case X must have less number of elements. The notation is $X \subsetneq Y$.

- *An Interesting Point: Note that empty set $\{\}$ is a subset of any set, why?*
- Ans: This is because even if we think what belongs to empty set also belongs to another set, this is always true, since empty set contains nothing. We sometimes call this *Vacuously True*!

- Let's do some examples

  Example 2A.2: Suppose we have following sets,

  $$A = \{a, b, c\}, \quad B = \{a, b, c\}, \quad C = \{b, c\} \text{ and } D = \{c\}$$

  Then we can see that $A = B$, but $A \neq C$ and also $B \neq C$ and also $C \neq D$.

  Also note $A \subset B$, and also $B \subset A$, $C \subset B$ and also $D \subset C$, and also $C \subsetneq B$ (*Question:* Is it correct to write $D \in C$, Ans: No, why?)

Sets

b. Set Operations

- Now we will see some concepts which we call *Set Operations*. Set operations mean we will combine two or more sets in different ways. Following are important set operations.
- **1. Union of two sets:** The union of two sets A and B is the set of elements that belong *either* set A *or* set B or *both* of the sets. The notation we will use is $A \cup B$. So this means

$$A \cup B = \{x : x \in A \text{ OR } x \in B\}$$

- **2. Intersection of two sets:** The intersection of two sets A and B is the set of elements that belong to *both* A and B. We will use the notation $A \cap B$. So

$$A \cap B = \{x : x \in A \text{ AND } x \in B\}$$

- **3. Difference between two sets:** The difference (sometimes also called relative difference) of A and B is the set of elements that *belong to* A but *not belong to* B. The notation is $A \setminus B$. We can write,

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

- **4. Product of two sets:** If A and B are sets, then the product of two sets is called *Cartesian product*. The *Cartesian product* of A and B is the *set* of all *ordered pairs* (a, b) such that $a \in A$ and $b \in B$. So using set building notation we can write,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- Note in the Cartesian Product *ordering* is important. For two sets A and B, $A \times B$ is not same as $B \times A$ (look at the next example).

Example 2A.3: Suppose we have following sets,

$$A = \{a, b, c\}, \quad B = \{a, b, c\}, \quad C = \{b, c\},$$
$$D = \{c\}, \quad E = \{1, 2\}$$

Now $A \cup B = \{a, b, c\}$, $C \cup D = \{b, c\}$, $C \cap D = \{c\}$, $B \setminus C = \{a\}$.

Let's think about Cartesian Products,

$$A \times E = \{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\} \text{ but}$$
$$E \times A = \{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

- Note that, *ordering matters* for the product of sets, so $(a, 1) \neq (1, a)$. So $A \times E \neq E \times A$.

Idea of the Universal Set and Complements

- Often (depending upon the problem) we have a *universal set*. The idea of the universal set is, it acts like a "*Universe*", this means everything belongs here. We usually denote the universal set with U (see example below)

$$U = \{a, b, c, 1, 2\},$$
$$A = \{a, b, c\}, \quad B = \{a, b, c\}, \quad C = \{b, c\} \quad D = \{c\}, \quad E = \{1, 2\}$$

  Note that all sets are subsets of the Universal Set U

- Now once we have a universal set, then we can find a complement of which is the remaining part of the Universal set. For example, take the set A from the last example, here $A \subset U$, then the complement set is

$$A^c = U \setminus A \tag{1}$$

  for complement sometimes there is another notation $\bar{A}$, I think **?** uses this notation.

- What is B^c and C^c?

Sets and Related Ideas
Set Operations

- We can have a visual understanding of union, intersection, complement and set difference using a diagram called *Venn diagram*.
- What is a Venn Diagram? The idea of the Venn diagram is to show these concepts visually with balls and squares (A famous mathematician, logician and philosopher Jonn Venn (1834 1923) came up with this idea!)
- Here are some Venn Diagrams for the concepts we have learned so far.

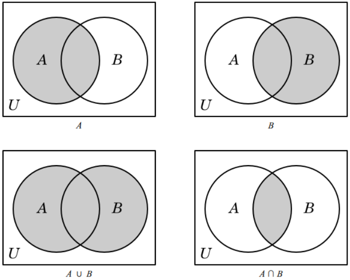


Figure 1: Venn diagram for $A \cup B$, $A \cap B$

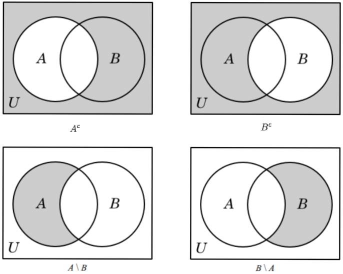


Figure 2: Venn diagram for A^c, B^c, $A \setminus B$, and $B \setminus A$

- The idea of Unions, Intersections, Cartesian Product and also Difference can be easily extended to more than two sets. In this case the idea is same, you just apply this similarly (this is the beauty of Math!)
- For example if we have a *sequence of sets* A_1, A_2, A_3, then for union and intersection we will have,

$$A_1 \cup A_2 \cup A_3 \text{ and } A_1 \cap A_2 \cap A_3$$

- For Cartesian Product we can also do

$$A_1 \times A_2 \times A_3$$

- And for difference we also have

$$A_1 \setminus A_2 \setminus A_3$$

- Think about some examples?

- Now we will learn some *laws / rules for sets when it comes to combining the operations like unions, intersections and difference*.
- I will just mention three laws, which are the most fundamental ones, but it is possible to construct more. Laws means these are some mathematical statements that we need to prove (careful : this is not definition, for definition / axioms there is no proof, we just take them as it is). In general in Math we need to prove statements which are laws / rules / theorems / corollary / propositions / lemma ... but we will not do any proof here, maybe in higher courses you will learn some how to do mathematical proof, here we will simply learn the rules and apply them!

Laws of Sets (Associative, Distributive and De-Morgan's Law)

- *Associative Law*

$$1.\ A \cup (B \cup C) = (A \cup B) \cup C$$
$$2.\ A \cap (B \cap C) = (A \cap B) \cap C$$

- *Distributive Law*

$$1.\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$2.\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- *Demorgan's Law*

$$1.\ (A \cap B)^c = A^c \cup B^c$$
$$2.\ (A \cup B)^c = A^c \cap B^c$$

- Although we won't do any proof here, but it's very important that we understand the laws properly.
- The first law says - if we first take the union of the sets B and C and then take the union of A with the result of $B \cup C$, then this will be same as first taking the union of A and B and then taking the union of $A \cup B$ with C.

- Can you interpret others?
- Let's do an example, Let's verify the first distributive law, Let $A = \{4, 5\}, B = \{3, 6, 7\}$, and $C = \{2, 3\}$.
- To verify the first part of the law, we find the left- and right-hand expressions separately:
- *Calculation for Left:* first calculate $B \cap C = \{3\}$ and then $A \cup (B \cap C)$, we get $A \cup (B \cap C) = \{4, 5\} \cup \{3\} = \{3, 4, 5\}$
- *Calculation for the right:* first calculate $A \cup B = \{3, 4, 5, 6, 7\}$ and then $A \cup C = \{2, 3, 4, 5\}$, now we do $(A \cup B) \cap (A \cup C)$ and here $(A \cup B) \cap (A \cup C) = \{3, 4, 5\}$

Power Set : The set of all possible subsets

- Now let's talk about the *Power Set*. Power set is actually *a set of sets*
- The power set of a set A, is a *set of of all possible subsets of A*. The notation for power set of A is $\mathcal{P}(A)$.

Example 2A.4: (Power Set)

- If we have a set $A = \{1, 2\}$, then $\mathcal{P}(A) = \Big\{\{\}, \{1\}, \{2\}, \{1, 2\}\Big\}$.
- If we have a set $A = \{1, 2, 4\}$, then

$$\begin{aligned}\mathcal{P}(A) &= \Big\{\{\}, \{1\}, \{2\}, \{1, 2\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\Big\} \\ &= \Big\{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{4\}, \{1, 4\}, \{2, 4\}, A\Big\}\end{aligned}$$

- If we have a set $B = \{a, b, c, d\}$, calculate $\mathcal{P}(B)$
- If we have a set $\Omega = \{H, T\}$, calculate $\mathcal{P}(\Omega)$
- If we have a set $\Omega = \{1, 2, 3, 4, 5, 6\}$, calculate $\mathcal{P}(\Omega)$

- So in simple words, a *power set is a set of sets* which has all the subsets that we can construct with the elements of A.
- Often a *set of sets* is called a *family of sets*.

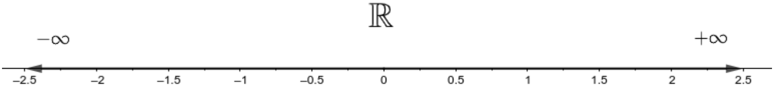
- There is a nice trick which we can use to count the number of elements in the power set if the given set is countable and finite. For example if A is a *countable and finite set** then $\mathcal{P}(A)$ will have $2^{(\text{number of elements in } A)}$
- For example, if $A = \{1, 2\}$, then the power set $\mathcal{P}(A)$ will have $2^2 = 4$ elements. Note that this matches with our earlier answer.
- Can you think about power set of $\mathbb{R}$, so this means can you think about $\mathcal{P}(\mathbb{R})$? (This is huge right but maybe an idea?)

*Question is what does countable mean in Mathematics? It means - we can *enumerate* the elements. A set can be countably finite, countably infinite and uncountable. Uncountable sets are always infinite, it is not possible to have uncountable but finite sets. Examples.....(on board)

Sets

c. Different Sets of Numbers

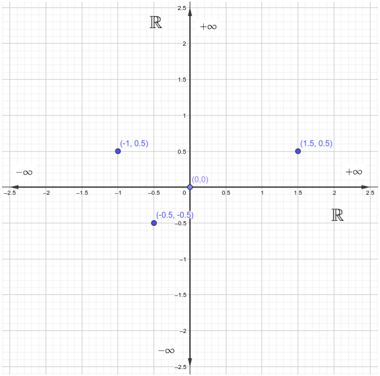
- There are different *sets of numbers* in mathematics.
  - *Set of Real numbers*, we use the notation $\mathbb{R}$ to denote this set. This set include all numbers that you can possibly think about (except complex numbers, which we don't need now). This is a huge set which is *uncountable* and of course *infinite*.
  - *Set of Natural numbers*, we use the notation $\mathbb{N}$. This set include all positive integer numbers $1, 2, 3, 4, \ldots,$. This is a *countable set* but an infinite set, so this is a *countably infinite* set (are you wondering what does countable mean?).
  - *Set of Integer numbers*, we use the notation $\mathbb{Z}$. This set include all the positive and negative integer numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. This is also a *countably infinite set*. Note that $\mathbb{N} \subsetneq \mathbb{Z}$
  - *Set of Rational numbers*, we use the notation $\mathbb{Q}$. This set include the numbers which can be written as fractions p/q, where p and q are both integers. This set has numbers like $2/3, 10/3$ and also all positive and negative integers are also part of this set (why?). This is also a *countably infinite set*. $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$
  - *Set of Irrational numbers*. Everything that is NOT Rational but in $\mathbb{R}$ is part of this set, for example $\sqrt{2}$, We can write this set with $\mathbb{R} \setminus \mathbb{Q}$.
  - This means we can write $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$
- The set of real numbers $\mathbb{R}$ can be also visualized in the numberline, here is the numberline that you are probably familiar with



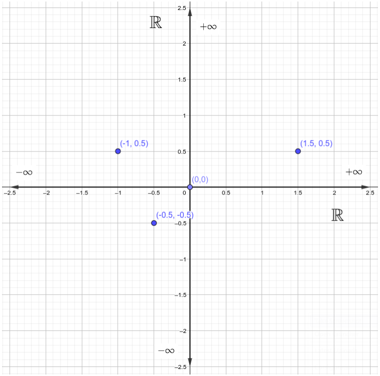
- At the center, we have the number 0 (this is often called the origin or center), at left the number goes to $-\infty$ and right it goes to ∞.
- We can point any number that belongs to $\mathbb{R}$ in the numberline. Here we showed only a few.
- We can also have different kinds of intervals in $\mathbb{R}$, which are also subsets of $\mathbb{R}$. For example we can construct following intervals (here a and b can be any number in $\mathbb{R}$)

Notation	Set description	Type	Picture
(a, b)	$\{x \mid a < x < b\}$	Open	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
(a, ∞)	$\{x \mid x > a\}$	Open	
$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
$(-\infty, b)$	$\{x \mid x < b\}$	Open	
$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

- Intervals like (a, b) is called *open intervals*, intervals like $[a, b]$ *closed intervals*, intervals like $(a, b]$ or $[a, b)$ are called *half-open* intervals.
- Recall the idea of Cartesian product? Can you think about the Cartesian product $\mathbb{R} \times \mathbb{R}$? Although it is impossible to write the set $\mathbb{R} \times \mathbb{R}$, but we can visualize it.
- Just put another numberline vertically on top of the horizontal one, then we will have something which is known as *Cartesian Coordinate* or *$x - y$ Coordinate* or *$x - y$ plane*
- Now here we have a horizontal axis, known as x-axis, and the vertical axis, known as y-axis.



- Here we can show any pair of numbers (x, y), where the first number is on the x-axis and the second is on the y-axis, and togther we can locate the point (x, y) on this $x - y$ plane.
- We have showed the center, which is at $(0, 0)$, and also other three points, $(-1, 0.5)$, $(1.5, 0.5)$ and $(-0.5, -0.5)$


$\mathbb{R}$
$+\infty$
(-1, 0.5)
(1.5, 0.5)
$-\infty$
$+\infty$
(0,0)
(-0.5, -0.5)
$\mathbb{R}$
$-\infty$

1. **Sets**
   - a. Definitions and Examples
   - b. Set Operations
   - c. Different Sets of Numbers

2. **Functions and Related Ideas**

3. **Math Recap - Counting Methods**
   - a. Multiplication Rule
   - b. Permutation
   - c. Combination

Functions and Related Ideas

Functions and Related Ideas

- If you have taken any math courses, whether in college or courses like MAT100 or MAT110, you have definitely seen functions.
- For example following are all examples of functions,
  - $f(x) = x^2$
  - $f(x) = 2x + 1$
  - $f(x) = 3x^3 + 2x^2 + 1$
- It is very easy to understand a function, there is always *an input* and and *an output*, and the function specifies this input-output relation. Important is for each input there is only one output, there cannot be more than one.
- Following picture might be helpful

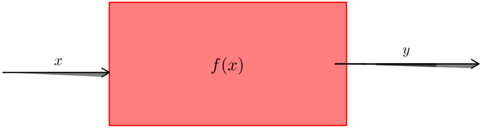


Figure 3: A function, we write $y = f(x)$ specifies a process here x can be viewed as an input and y or $f(x)$ is the output.

Functions and Related Ideas

- Formal definition of definition is similar, it just makes the definition more precise.

Definition 2A.5: (Function)

Given any two sets A and B, a function $f : A \to B$ is a *mapping* between the elements of A and B such that the following condition is satisfied

- For every element of A there is a unique element in B.

In this case, the set A is called the *domain* of the function f and B is called the *codomian* of the function.

- It is important to mention that although in the definition we wrote one condition, that is *"For every element of A there is a unique element in B."*, this actually means two points,
  - First, Since we are saying "For *every* element of A...", the word "every" here automatically means all elements of A needs to be used for mapping .
  - Second, when we say for each element of A, there must be a unique element in B. This means it will never happen that a single point from A is mapped to two different points in B.
- We won't go to more details here, please MAT-100 / College Math notes if you need to brush up!

Math Recap - Counting Methods

a. Multiplication Rule

Counting Methods - Multiplication rule

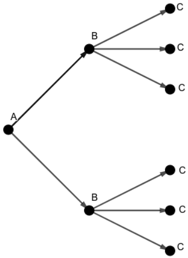
- In this chapter we will look into different counting principles, consider following examples,
- **Routes between Cities** Suppose that there are 3 different routes from city A to city B and 5 different routes from city B to city C. How many different ways we can go from A to C via B?
- **Choosing President and Vice-President:** Suppose a political party consists of 25 members and that a president and a vice-president are to be chosen from the membership. First president will be chosen and then the vice president. How many ways we can fill this two positions?
- **Choosing members** Suppose a political party consists of 25 members and we need to select any two members. How many ways we can select two members?
- We will start with Multiplication rule, which is the basis of other techniques we will learn in this section. In the probability theory chapter this will help us to count the number of elements in the sample space or any event!

Counting Methods - Multiplication rule

- The *multiplication rule* helps to solve some counting problems when we have a process with more than one parts or steps. With this we can count *how many ways the entire process can be performed*.
- We will explain the method with some simple examples and drawing a diagram called tree diagram...

Counting Methods - Multiplication rule

Example 2A.6:Multiplication Rule

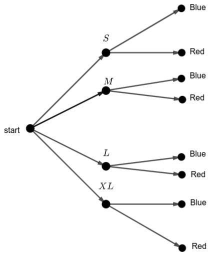


- Suppose we have three cities, A, B and C. We need to go from A to C via B?
- If there are 2 ways we can go from A to B (call them *path-1 and path-2*) and 3 ways we can go from B to C (call them *path-3, path-4 and path-5*), question is *how many ways we go from A to C via B*?
- First note that the process has two parts, the first part we have 2 *possible ways* and in the second part we have 3 *possible ways*.
- So the whole process can be performed in $2 \times 3 = 6$ possible ways.
- For the multiplication problems, the tree diagram (figure below) could be useful to visualize.
- We can list them using set (note that this is actually Cartesian Product of ....?)

$$\{(\text{Path-1, Path-3}), (\text{Path-1, Path-4}), (\text{Path-1, Path-5}),$$
$$(\text{Path-2, Path-3}), (\text{Path-2, Path-4}), (\text{Path-2, Path-5})\}$$

Counting Methods - Multiplication rule

Example 2A.7:(Multiplication Rule)



- Suppose a retail store sells *windbreaker jackets* in *sizes* small (S), medium (M), large (L), and extra large (XL). All are available in *color* "blue" or "red". If a buyer wants to buy then *how many options/choices are available for the buyer*?

- Applying multiplication rule, we get in total there are $4 \times 2 = 8$ possible choices. Again we can write them using sets

$$\{(S, \text{Blue}), (S, \text{Red}), (M, \text{Blue}),$$
$$(M, \text{Red}), (L, \text{Blue}), (L, \text{Red}),$$
$$(XL, \text{Blue}), (XL, \text{Red})\}$$

Counting Methods - Multiplication rule

- Now we can see the formal definition,

Multiplication Rule

If a procedure or a process consists of k *independent parts* (where $k \geq 2$) and the i^{th} part can be performed in n_i possible ways (where $i = 1, 2, \ldots, k$), then the entire process can be performed in $n_1 \times n_2 \times \ldots, \times n_k$ possible ways.

- Here i^{th} part can be performed in n_i possible ways means,
  - when $i = 1$, 1^{st} process can be performed with n_1 ways
  - when $i = 2$, 2^{nd} process can be performed with n_2 ways
  - $\vdots$
  - when $i = k$, k^{th} process can be performed with n_k ways
- *independent parts* means one part is not dependent on the other.

Counting Methods - Multiplication rule

- More examples,

  Example 2A.8:(Multiplication Rule)

- **Question:** Suppose a coin is tossed 6 times, how many possible outcomes are there?

- **Ans:** Actually there will be $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 36$ possible outcomes. Can you list all possible outcomes. For example one outcomes is $HTTHHH$ (can you think about the tree diagram here?). The Cartesian product of the set $\{H, T\}$ six times will give you all possible outcomes. Can you draw the tree diagram (yes but this is cumbersome)?

  Example 2A.9: (Multiplication Rule)

- **Question:** Suppose we have a 3 digit combination lock where each digit can be from 0 to 9. How many possible combination locks we can set?

- **Answer:** There will be $10 \times 10 \times 10 = 10^3 = 1000$ possible combination locks. Can you draw the tree diagram (yes but this is cumbersome)?

Math Recap - Counting Methods

b. Permutation

Counting Methods - Permutation

- So multiplication problem is quite simple, in the next problem we consider one special case of the problem where *which comes first or second matters, or we say ordering matters*.
- For the lock problem, first note we can think about the same problem in the following way,
- think about *three empty boxes* and then we want to know *how many possible ways we can fill the boxes?*
- For the first box we have 10 possible options, for the second we also have 10, and for the third we also have 10. This gives the following picture

$$\boxed{10} \times \boxed{10} \times \boxed{10}$$

- This means we have $10 \times 10 \times 10 = 1000$ possible options for locks.
- Note this is the same problem but we are thinking now with boxes, rather than tree diagram.

Counting Methods - Permutation

- Now consider the same problem, but *suppose we don't want any repetition, this means the same digit cannot appear more than once*.
- So we want all three digits to be different. For example we don't want to count $0, 0, 1$ or $1, 1, 1$ as possible count.
- We can solve this problem using the box idea.
- For the first place we have 10 digits, for the second place we have 9 digits, and for the third we have 8 digits left, this means we have now

$$\boxed{10} \times \boxed{9} \times \boxed{8}$$

- So $10 \times 9 \times 8 = 720$ possible combinations.
- So problem solved!
- Note we still solved this problem as a multiplication problem.

Counting Methods - Permutation

- The last problem can be also understood *as a ordering problem*, roughly this means a problem where *ordering matters*.
- In general for any ordering problem, we don't have to think about the combination lock, the idea is if we have 10 digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we ask *how many ways we can order any 3 digits*.
- And the answer is same $10 \times 9 \times 8 = 720$.
- What if we order 10 objects such that ordering matters, then the answer is $10 \times 9 \times 8 \times \ldots \times 1$
- Which in short we write, 10! or we say *10 factorial*.
- Note for any non-negative integers n,

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$$

- For example $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$, and $4! = 4 \times 3 \times 2 \times 1 = 24$, and $3! = 3 \times 2 \times 1 = 6$.
- We define $0! = 1$,

Counting Methods - Permutation

- So with this factorial idea we can now write the total number of orderings for three digits out of 10, as

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8$$

- which is the famous *formula for permutation* and we write this as ${}^{10}P_3$, in general

Definition 2A.10: (Permutation)

If we have n objects and we want to order k of them, then the total number of orderings is given by

$${}^{n}P_k = \frac{n!}{(n-k)!}$$

- *Side Note*: *Ques: What does it mean we say "ordering matters?"*, Ans: this means when we are counting we are treating $1, 2, 3$ and $2, 1, 3$ as a separate count, not as a same count.
- Also note the word "permutation" in English just means "rearrangement". For example if we have three letters a,b,c then a another permutation (or ordering) is b,a,c. So when we ask total number of permutations, this is same asking total number of arrangements or total number of orderings.

Counting Methods - Permutation

- So , in general if someone asks you "if we have n objects then how many ways we can order k of them if we pick one at a time?" the answer is

$$^{n}P_k = \frac{n!}{(n-k)!}$$

- Or you can think with the boxes, in this case you can think n objects and k empty boxes and we are trying to fill them one by one.
- And as you already understood, if we have n objects and n boxes, then $^{n}P_n = n!$ (here we used $0! = 1$) This means $n!$ gives the total number of orderings when we want to order n objects and we have n empty boxes.
- Let's do some examples.

Counting Methods - Permutation

- Ques: If we have 5 letters a, b, c, d, e, then
  - a) How many ways we can order them?
  - b) How many ways we can order 3 of them (taking one at a time)?
- The answer of a) is $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.
- To answer b), we can follow one of the two approaches
  - **Empty box approach (perhaps this is more intuitive):** We have 3 empty boxes, so for the first box we have 5 options, for the second box we have 4 and for the third we have 3. So in total we have $5 \times 4 \times 3 = 60$. So there are 60 possible ways we can order them
  - **Directly applying the formula:** Since this is a direct ordering problem we can apply the permutation formula, ${}^5P_3 = \frac{5!}{(5-3)!} = 60$.
- Here are all 60 permutations if we pick 3 letters out of 5.

a, b, c	b, a, c	c, a, b	a, c, b	b, c, a	c, b, a
a, b, d	b, a, d	d, a, b	a, d, b	b, d, a	d, b, a
a, b, e	b, a, e	e, a, b	a, e, b	b, e, a	e, b, a
a, c, d	c, a, d	d, a, c	a, d, c	c, d, a	d, c, a
a, c, e	c, a, e	e, a, c	a, e, c	c, e, a	e, c, a
a, d, e	d, a, e	e, a, d	a, e, d	d, e, a	e, d, a
b, c, d	c, b, d	d, b, c	b, d, c	c, d, b	d, c, b
b, c, e	c, b, e	e, b, c	b, e, c	c, e, b	e, c, b
b, d, e	d, b, e	e, b, d	b, e, d	d, e, b	e, d, b
c, d, e	d, c, e	e, c, d	c, e, d	d, e, c	e, d, c

- It's important that you understand the phrase *"ordering matters"* in the permutation problem,
- In our example this means when we pick three letters we take all possible permutations when we do the counting, for example, take the first row where we have different permutations of the letter a, b, c. In total there are $3 \times 2 = 6$ permutations,.... ordering matters means when we count, we count all 6 of them.
- Similarly in every row we have 6 permutations of three letters and when we count we count all of them.
- **Question:** *What if we treat them as a single count?*

Math Recap - Counting Methods

c. Combination

Counting Methods - Combination

- The answer to last question lies in the idea of *combination problem*. In the combination problem we use the idea of *"selection"*, rather than ordering, and we count all permutations of the same objects as one count.
- For example, in the last letter problem, if we ask *"how many ways we can select* 3 *letters out of* 5 *letters", the answer is* 10 (notice the word "select" here)
- Here "select" means, we are just selecting and we don't care about orders now.
- Now how did we get 10? Just count one for each row where we show all possible permutations. Since there are 10 rows we have 10 possible combinations. We can reach this by dividing the excess count, here is the formal definition,

Counting Methods - Combination

Definition 2A.11: (Combinations)

If we have a set of n elements. Each *subset* of size k chosen from this set is called a *combination of n elements taken k at a time*. We denote the number of *distinct* such combinations by the symbol ${}^{n}C_{k}$. And we can count this number by

$$ {}^{n}C_{k} = \frac{{}^{n}P_{k}}{k!} = \frac{n!}{k!(n-k)!} $$

- There is another way to understand the formula, the idea is let's think in this case the number of permutations being constructed in two steps or two parts.
- *Part 1:* How many ways we can select k elements from n, this is ${}^nC_k$ (this is the selection part)
- *Part 2* How many ways those k elements can be arranged / ordered within themselves. This is $k!$
- Now if we apply the multiplication rule and we get the total number of orderings,

$$^nP_k = {}^nC_k \times k!$$

- and from here we get our formula

$$^nC_k = \frac{^nP_k}{k!}$$

Counting Methods - Combination

- Also note we can say - the number of distinct subsets of size k that can be chosen from a set of size n is ${}^nC_k$.
- Or if someone asks you *"how many ways you can select* k *objects from* n*?"*, then the answer is ${}^nC_k = \frac{{}^nP_k}{k!} = \frac{n!}{k!(n-k)!}$
- There is another notation for the combination and that is $\left(\begin{array}{c} n \\ k \end{array} \right)$
- So $\left(\begin{array}{c} n \\ k \end{array} \right)$ is same as ${}^nC_k$, it means *"how many distinct ways we can select* k *objects from* n*?"*

There is a very useful application of ${}^nC_k$, we this call *Binomial Theorem*, here is the theorem.

Theorem 2A.12: (Binomial Theorem.)

For all numbers x and y and each positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k} y^k.$$

where $\left(\begin{array}{c} n \\ k \end{array} \right)$ is same as ${}^nC_k$, so this is possible number of combinations of k objects out of n.
In this case this is also known as *Binomial co-efficient*